Notes on the Poincaré Lemmas

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1 The Poincaré Lemma for de Rham Cohomology

We only consider de Rham cohomology in this section. Our goal is to prove the following statement:

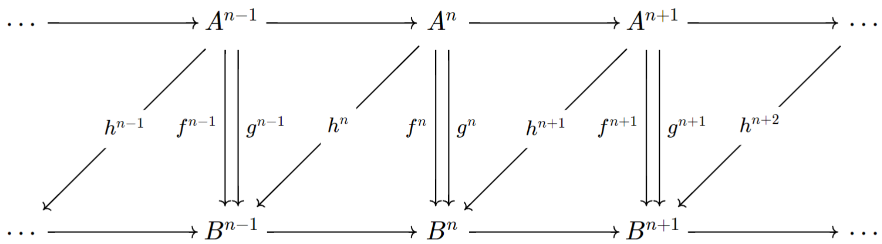
Proposition 1.1
$H^*(M \times \mathbb{R}) \simeq H^*(M)$. To find the isomorphism, let $\pi : M \times \mathbb{R} \to M$ the projection on the first factor, and $s : \mathbb{R}^n \to M \times \mathbb{R}$ be a section, namely:

$$\pi(x,t) = x$$
$$s(x) = (x,r),\ r \in \mathbb{R}$$

These two maps induce inverse isomorphisms on de Rham cohomology.

It is safe to assume s to be the zero section, which means $s(x) = s(x,0)$. We denote the induced maps on chain level by π^*, s^*. It is trivial that $s^* \circ \pi^* = 1$, so it remains to show that $\pi^* \circ s^*$ induces identity map on cohomology. Now we need a tool from algebraic topology.

Definition 1.2 (Chain Homotopy)
Let A^* and B^* be chain complexes, f, g be chain maps. A chain homotopy from f to g is a collection of maps $h^n : A^n \to B^{n-1}$, such that $f - g = dh + hd$ (indexes omitted).



If the chain homotopy exist, we say that f and g are chain homotopic.

If f and g are chain homotopic, then they induce the same map on cohomology, since $dh - hd$ induces zero map. So to prove $\pi^* \circ s^*$ induces identity map on cohomology, we need to find the chain homotopy from 1 to $\pi^* \circ s^*$.

To construct the chain homotopy, we claim that every form on $M \times \mathbb{R}$ is uniquely a linear combination of the following two types of forms:

$$\text{(I)} \quad (\pi^*\phi) f(x,t)$$
$$\text{(II)} \quad (\pi^*\phi) f(x,t)\ dt,$$

where ϕ is a form on M. These are the part without dt and the part with dt. To prove the claim, first notice this is true when $M = \mathbb{R}^n$, then in general case, use the fact that $M \times \mathbb{R}$ is locally $\mathbb{R}^n$. Now define $K^q : \Omega^q(M \times \mathbb{R}) \to \Omega^{q-1}(M \times \mathbb{R})$ by

$$\text{(I)} \quad (\pi^*\phi) f(x,t) \mapsto 0$$
$$\text{(II)} \quad (\pi^*\phi) f(x,t)\ dt \mapsto (\pi^*\phi) \int_0^t f.$$

Direct computation shows that $1 - \pi^* \circ s^* = (-1)^{q-1}(dK - Kd)$. Assign appropriate sign to each K^q, we obtain chain homotopy from 1 to $\pi^* \circ s^*$. Thus proposition 1.1 is proved.

Corollary 1.3 (Poincaré Lemma)

$$H^*(\mathbb{R}^n) \simeq H^*(\mathbb{R}^{n-1}) \simeq \cdots \simeq H^*(point)$$

Corollary 1.4 (Homotopy Axiom for de Rham Cohomology)
Homotopic maps induce the same map in de Rham cohomology.

Proof: Recall that a homotopy between two maps f and g from M to N is a map $F : M \times \mathbb{R} \to N$ such that

$$F(x,t) = \begin{cases} f(x) & \text{for } t \geqslant 1 \\ g(x) & \text{for } t \leqslant 0. \end{cases}$$

Let s_0 and $s_1 : M \to M \times \mathbb{R}$ be the 0-section and 1-section, then

$$f = F \circ s_1$$
$$g = F \circ s_0.$$

Thus

$$f^* = s_1^* \circ F^* = (\pi^*)^{-1} \circ F^* = s_0^* \circ F^* = g^*$$

■

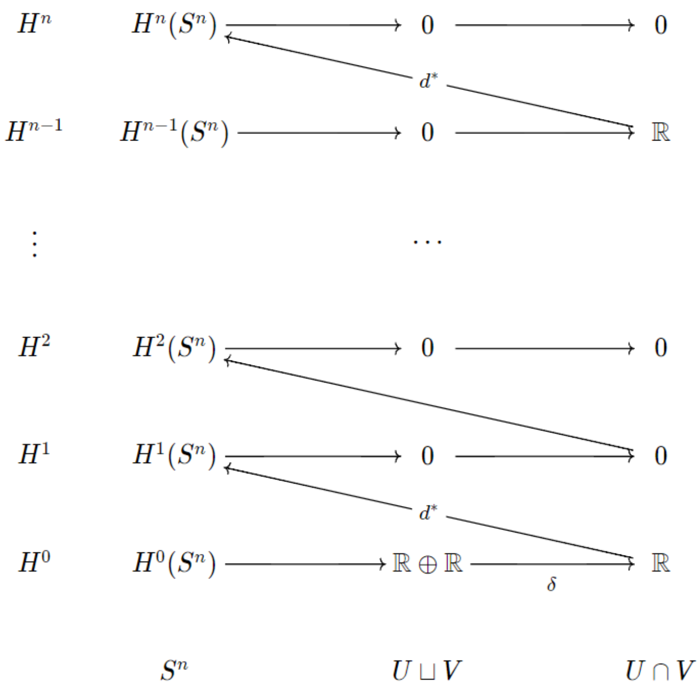
This means de Rham cohomology is a homotopy invariant.

We can compute $H^*_{dR}(S^n)$ with the help of Poincaré lemma. The process is basically the same as calculating singular homology of S^n.

Corollary 1.5

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{for } * = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We do it by induction and assume that it is true for S^{n-1}. Cover S^n by two open sets U and V where U is slightly larger than the northern hemisphere and V slightly larger than the southern hemisphere. Since $U \cap V$ can deform into S^{n-1}, we have Mayer-Vietoris sequence:



It is then obvious that $H^k(S^n) = 0$ for $k = 2, \cdots, n-1$, and $H^n(S^n) = \mathbb{R}$. S^n is connected, so $H^0(S^n) = \mathbb{R}$. Now to calculate $H^1(S^n)$, notice $\delta : (\omega, \tau) \to \tau - \omega$ is surjective, so $d^* : \mathbb{R} \to H^1(S^n)$ is zero map, which leads to $H^1(S^n) = 0$. ■

From the Mayer-Vietoris sequence, we also know that d^* send the generator of $H^{n-1}(S^{n-1})$ to the generator of $H^n(S^n)$. Example 2.6 in GTA82 shows that the generator of $H^1 S^1$ is a bump 1-form, sending this through repeatedly d^*, we get a bump n-form on S^n that generates $H^n(S^n)$.

2 The Poincaré Lemma for Compactly Supported Cohomology

In this section, we adapt proposition 1.1 to compactly supported cohomology.

Proposition 2.1

$$H_c^{*+1}(M \times \mathbb{R}) \simeq H_c^*(M).$$

The shift in dimension makes compactly supported cohomology not a homotopy invariant.

Consider projection $\pi : M \times \mathbb{R} \to M$. The pullback is not well defined for compactly supported cohomology, so we define the push-forward map $\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$ called **integration along the fiber**, defined as follows. First note that a compactly supported form on $M \times \mathbb{R}$ is again a linear combination of these two type of forms:

$$\begin{aligned} &\text{(I)} \quad (\pi^*\phi) f(x,t) \\ &\text{(II)} \quad (\pi^*\phi) f(x,t)\ dt, \end{aligned}$$

where ϕ is a (not necessarily with compact support) form on the base M, and $f(x,t)$ is a function with compact support. We define π_* by

$$\begin{aligned} &\text{(I)} \quad (\pi^*\phi) f(x,t) \mapsto 0 \\ &\text{(II)} \quad (\pi^*\phi) f(x,t)\ dt \mapsto \phi \int_{-\infty}^{\infty} f(x,t) dt. \end{aligned}$$

Direct calculation shows that $\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$ is a chain map, so it induce map in cohomology $\pi_* : H_c^*(M \times \mathbb{R}) \to H_c^{*-1}(M)$ (still denoted by π_*). To find an inverse for π_*, let $e = e(t)\ dt$ be a compactly supported 1-form on $\mathbb{R}$ with total integral 1 and define

$$\begin{aligned} e_* : \Omega_c^*(M) &\to \Omega_c^{*+1}(M \times \mathbb{R}) \\ \phi &\mapsto (\pi^*\phi) \wedge e. \end{aligned}$$

e is also a chain map, so it induces a map in cohomology, which we still denote by e_*. Obviously $\pi_* \circ e_* = 1$ on $\Omega_c^*(M)$. To prove $e_* \circ \pi_* = 1$, we again construct a chain homotopy. Define $K : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^*(M \times \mathbb{R})$ by

$$\begin{aligned} &\text{(I)} \quad (\pi^*\phi) f(x,t) \mapsto 0 \\ &\text{(II)} \quad (\pi^*\phi) f(x,t)\ dt \mapsto (\pi^*\phi) \left\{ \int_{-\infty}^{t} f - \left(\int_{-\infty}^{t} e \right) \left(\int_{-\infty}^{\infty} f \right) \right\}, \end{aligned}$$

where all integrals are integrations along the fiber $\mathbb{R}$. It can be shown that

$$1 - e_* \circ \pi_* = (-1)^{q-l}(dK - Kd),$$

thus proposition 2.1 is proved.

Corollary 2.2 (Poincaré Lemma for Compact Supports)

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } * = n \\ 0 & \text{otherwise.} \end{cases}$$

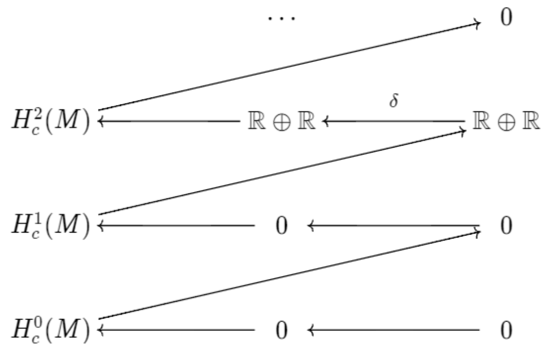
Here the isomorphism $H_c^n(\mathbb{R}^n) \simeq \mathbb{R}$ is given by iterated π_*, i.e., by integration over $\mathbb{R}^n$. To find a generator for $H_c^n(\mathbb{R}^n)$, send constant function 1 on one-point set by e_* for n times, which gives

$$e_1(x_1)\ dx_1 \wedge \cdots \wedge e_n(x_n)\ dx_n = \alpha(x)\ dx_1 \cdots dx_n$$

with $\int_{\mathbb{R}^n} \alpha(x)\ dx_1 \cdots dx_n = 1$. So the generator is a bump form.

Exercise: Compute the cohomology groups $H^*(M)$ and $H^*_c(M)$ of the open Möbius strip M.

Solution: $H^*(M) = H^*(S^1)$ since there exists a deformation retraction. Now we compute the compactly supported cohomology. Let $U, V \subseteq M$ be two rectangular strips that intersect at both ends, which are the coordinate charts of M. We have $U \cap V = A_1 \sqcup A_2$ and we can safely assume A_1 is the only place where the transition function is orientation reversing. Apply the Mayer-Vietoris sequences, we get



We claim that δ is an isomorphism, which implies $H^*(M) = 0$ for all dimensions.

Recall that

$$\delta : (\omega_1, \omega_2) \mapsto (-j_{U,*}(\omega_1, \omega_2),\ j_{V,*}(\omega_1, \omega_2))$$

where $\omega_i \in \Omega^2_c(A_i)$ and j_U, j_V are inclusion maps. Endow $A_1 \sqcup A_2$ with the orientation of U, then A_1 has different orientation with V, and A_2 has the same orientation with V. As a result, we have $\delta(\omega_1, \omega_2) = (-\omega_1 - \omega_2, -\omega_1 + \omega_2)$ (isomorphism $H^n_c(\mathbb{R}^n) \simeq \mathbb{R}$ is needed if we want a strict argument). Thus δ is an isomorphism.